方向邊際生產力與Meta-DEA
(Directional Marginal Productivity and Meta-DEA)

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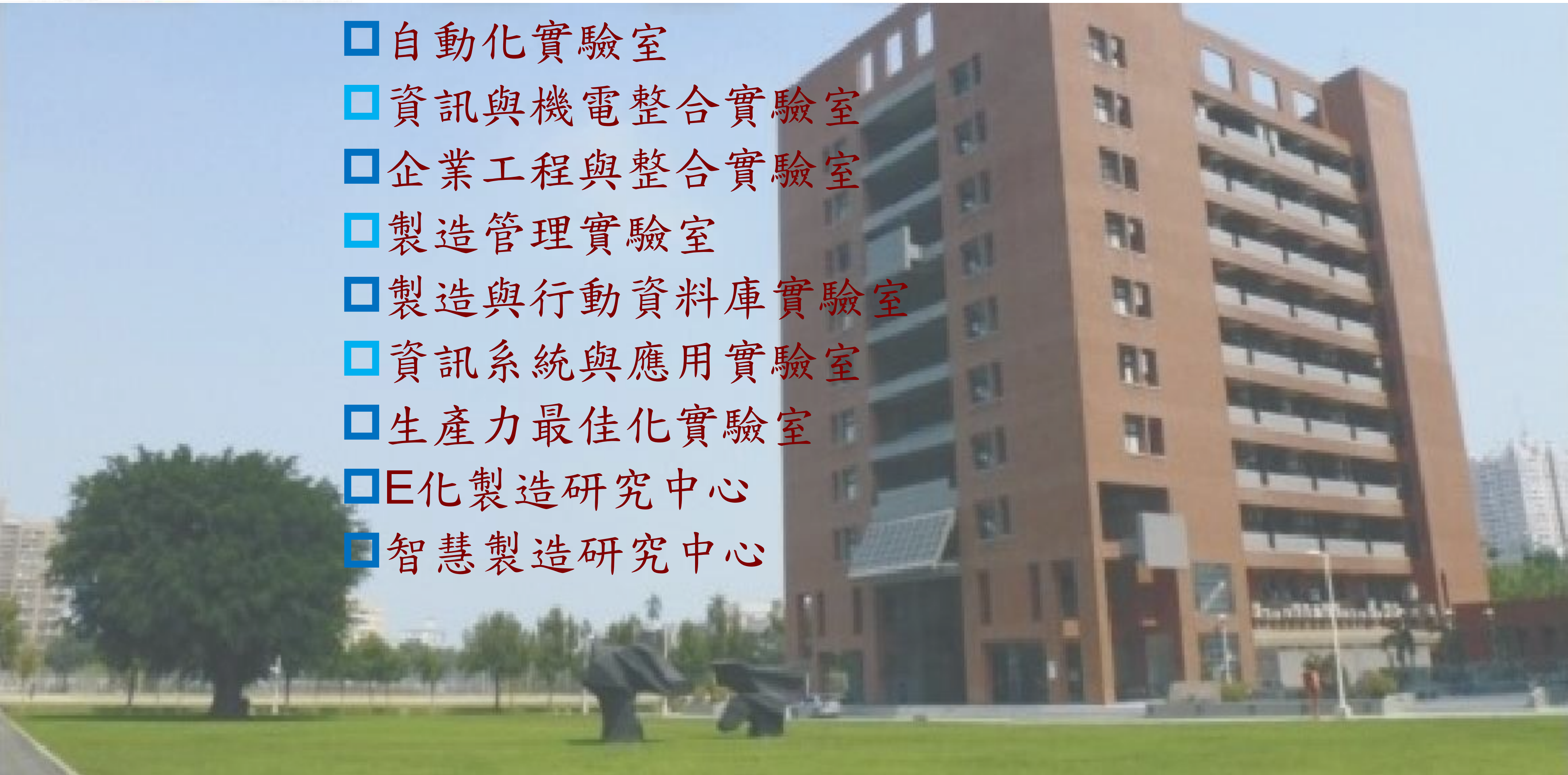

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Outline

- Background
- Directional Marginal Productivity (DMP)
- Meta-DEA
- Proactive DMP for Business Shutdown Decision

Directional Marginal Productivity

Lee, Chia-Yen, 2017. Directional Marginal Productivity: A Foundation of Meta-Data Envelopment Analysis. Journal of the Operational Research Society, 68 (5), 544-555.

Directional Marginal Productivity

- Introduction
- Single-Output Marginal Productivity
- DEA as Convex Nonparametric Least Squares
  - Convex Nonparametric Least Squares (CNLS)
  - Stochastic nonparametric envelopment of data (StoNED)
- Directional Distance Function (DDF)
- Directional Marginal Productivity by DDF
- Direction towards Marginal Profit Maximization
- Numerical Illustrations
- Conclusion

Directional Marginal Productivity

Background

- Multi-product differential characteristic of the production function generally is useful for production economics.
  - elasticity measures, marginal rates of substitution, shadow price
- Data envelopment analysis (DEA)using a piece-wise linear frontier forms a polyhedral set representing production technologies and thus is not differentiable.
- Podinovski and Førsund (2010) gave an explicit definition of differential characteristics on a non-differentiable efficient frontier and proposed a directional-derivative approach to calculate elasticity measures without any simplifying assumptions.

Motivation

- This study, which promotes Podinovski and Førsund's (2010) approach, uses directional distance function (Chambers et al., 1996) to develop multi-output directional marginal product.

Directional Marginal Productivity

Marginal Rate for Single Output

- Marginal Productivity $MP_A = \left.\frac{\partial f(x)}{\partial x}\right|_{X_A}$
- Podinovski and Førsund (2010) proposed a directional derivative technique to assess the marginal product (MP) of a nondifferentiable efficient frontier constructed by the DEA estimator.
- Marginal rate approaching from the right side

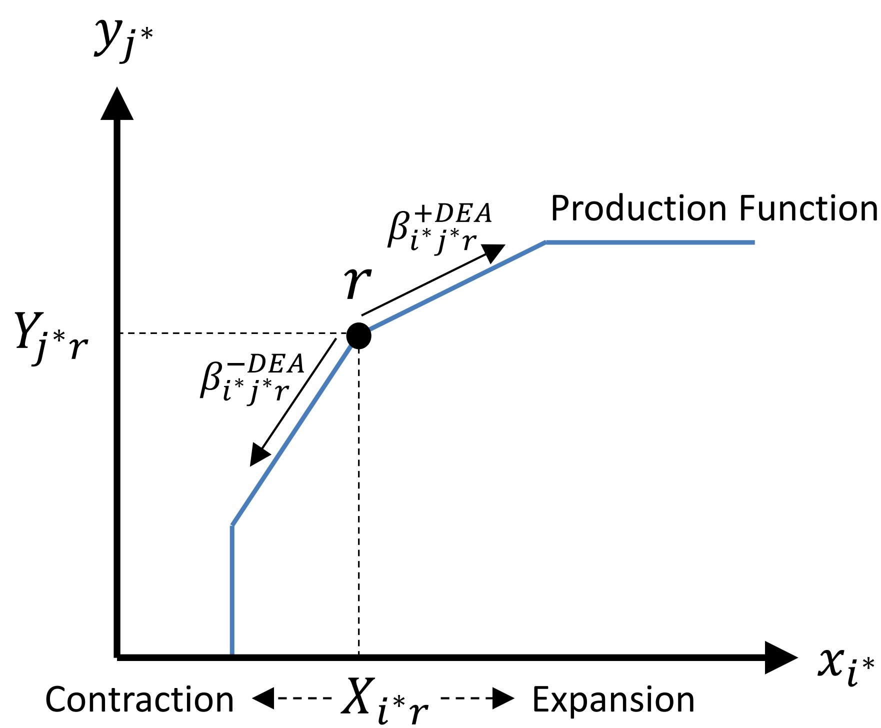
$$\frac{\partial Y_{j^*r}}{\partial X_{i^*r}} = \beta_{i^*j^*r}^{+DEA} = \text{Min } v_{i^*}$$

s.t. $\sum_i v_i X_{ir} - \sum_j u_j Y_{jr} + u_0 = 0$

$\sum_i v_i X_{ik} - \sum_j u_j Y_{jk} + u_0 \geq 0, \forall k$

$u_{j^*} = 1$

$v_i, u_j \geq 0$, u_0 is free



- from the left side

$$\frac{\partial Y_{j^*r}}{\partial X_{i^*r}} = \beta_{i^*j^*r}^{-DEA} = \text{Max } v_{i^*}$$

Directional Marginal Productivity

- How to validate the P&F's MP formulation?
  - Formulation

$$\frac{\partial Y_{j^*r}}{\partial X_{i^*r}} = \beta_{i^*j^*r}^{+DEA} = \text{Min } v_{i^*}$$

s.t. $\sum_i v_i X_{ir} - \sum_j u_j Y_{jr} + u_0 = 0$

$\sum_i v_i X_{ik} - \sum_j u_j Y_{jk} + u_0 \geq 0, \forall k$

$u_{j^*} = 1$

$v_i, u_j \geq 0$, u_0 is free

- The coefficient of independent variables in regression model represents the MP!

CNLS

- CNLS addresses the limitations of SFA and DEA
  - DEA: ignores random noise thus sensitive to outlier
  - SFA: requires specification of the functional form

- CNLS can be traced to the seminal work of Hildreth (1954) and was popularized by Kuosmanen (2008) as a powerful tool for describing the average behavior of observations.

$$\varepsilon_k^{CNLS} = \operatorname{argmin} \ \sum_k \varepsilon_k^2$$

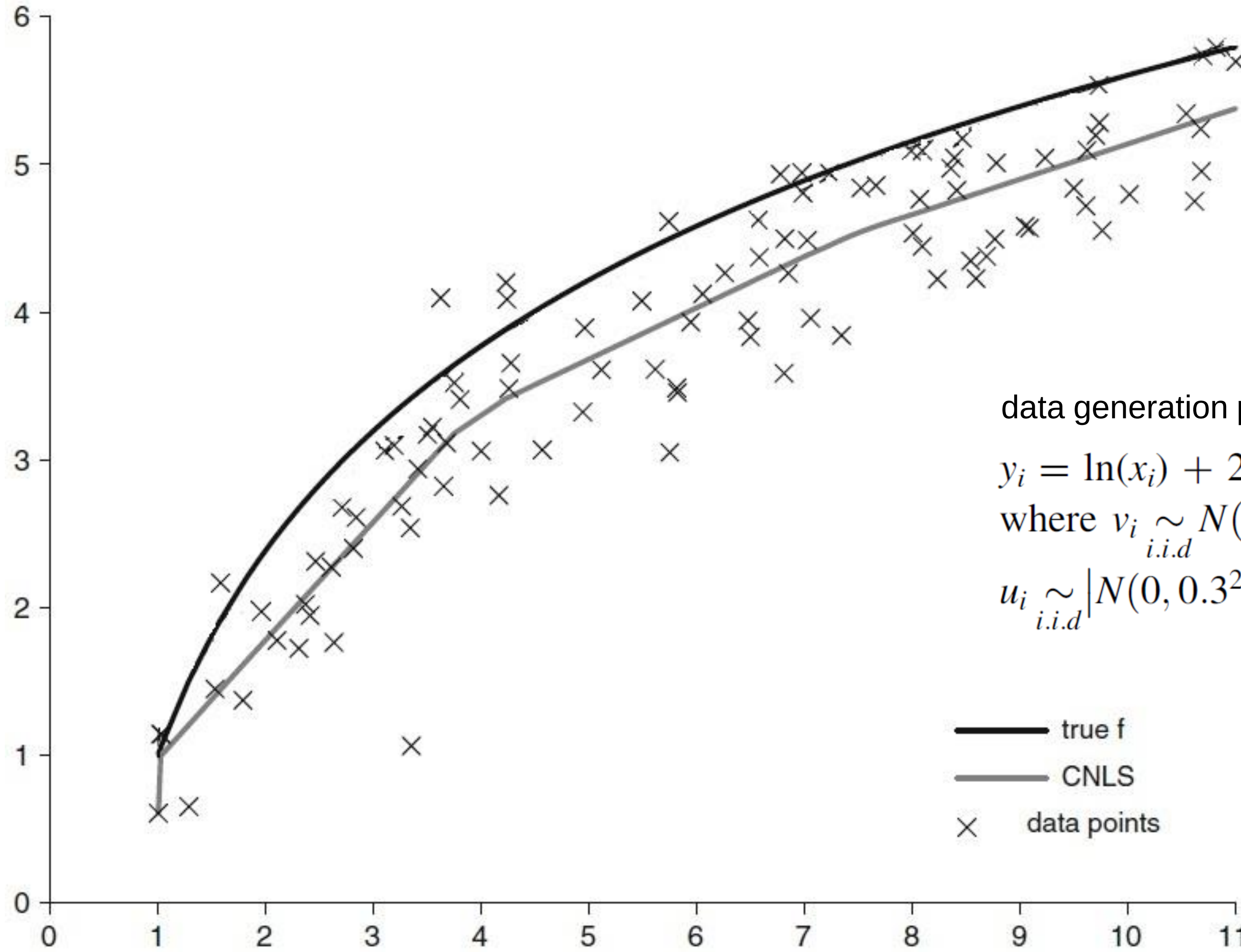
$$\text{s.t. } Y_k = \alpha_k + \sum_i \beta_{ik} X_{ik} + \varepsilon_k, \forall k \quad \text{(regression equation)}$$

$$\alpha_k + \sum_i \beta_{ik} X_{ik} \le \alpha_h + \sum_i \beta_{ih} X_{ik}, \forall k, \forall h \quad \text{(concavity constraint)}$$

$$\beta_{ik} \ge 0, \forall i, k \quad \text{(monotonicity constraint)}$$

- Shortcomings
  - Multiple solution (Kuosmanen and Kortelainen, 2012)
  - Computational burden (Lee et al., 2013)

Directional Marginal Productivity



data generation process (DGP)

$$y_i = \ln(x_i) + 2 + v_i - u_i,$$
where $v_i \underset{i.i.d}{\sim} N(0, 0.6^2)$ and
$$u_i \underset{i.i.d}{\sim} |N(0, 0.3^2)|n$$

Kuosmanen and Kortelainen (2012)

Directional Marginal Productivity

POLab

- StoNED (Kuosmanen and Kortelainen, 2012)
  - a new frontier estimation framework that combines the virtues of both DEA and SFA in a unified approach to frontier analysis.

$$\varepsilon_k^{CNLS} = \text{argmin} \ \sum_k \varepsilon_k^2$$

$$\text{s.t. } Y_k = \alpha_k + \sum_i \beta_{ik} X_{ik} + \varepsilon_k, \forall k \ \text{(regression equation)}$$

$$\alpha_k + \sum_i \beta_{ik} X_{ik} \le \alpha_h + \sum_i \beta_{ih} X_{ik}, \forall k, \forall h \ \text{(concavity constraint)}$$

$$\beta_{ik} \ge 0, \forall i, k \ \text{(monotonicity constraint)}$$

- Residual decomposition (stochastic)
  - Composite error: $\varepsilon_k = v_k - u_k$ (Aigner et al., 1977; Greene, 2008)
    - v_k: normally distributed noise term $v_i \underset{i.i.d}{\sim} N(0, \sigma_v^2)$
    - u_k: half-normal inefficiency term $u_i \underset{i.i.d}{\sim} |N(0, \sigma_u^2)|$

Directional Marginal Productivity

StoNED (Kuosmanen and Kortelainen, 2012)

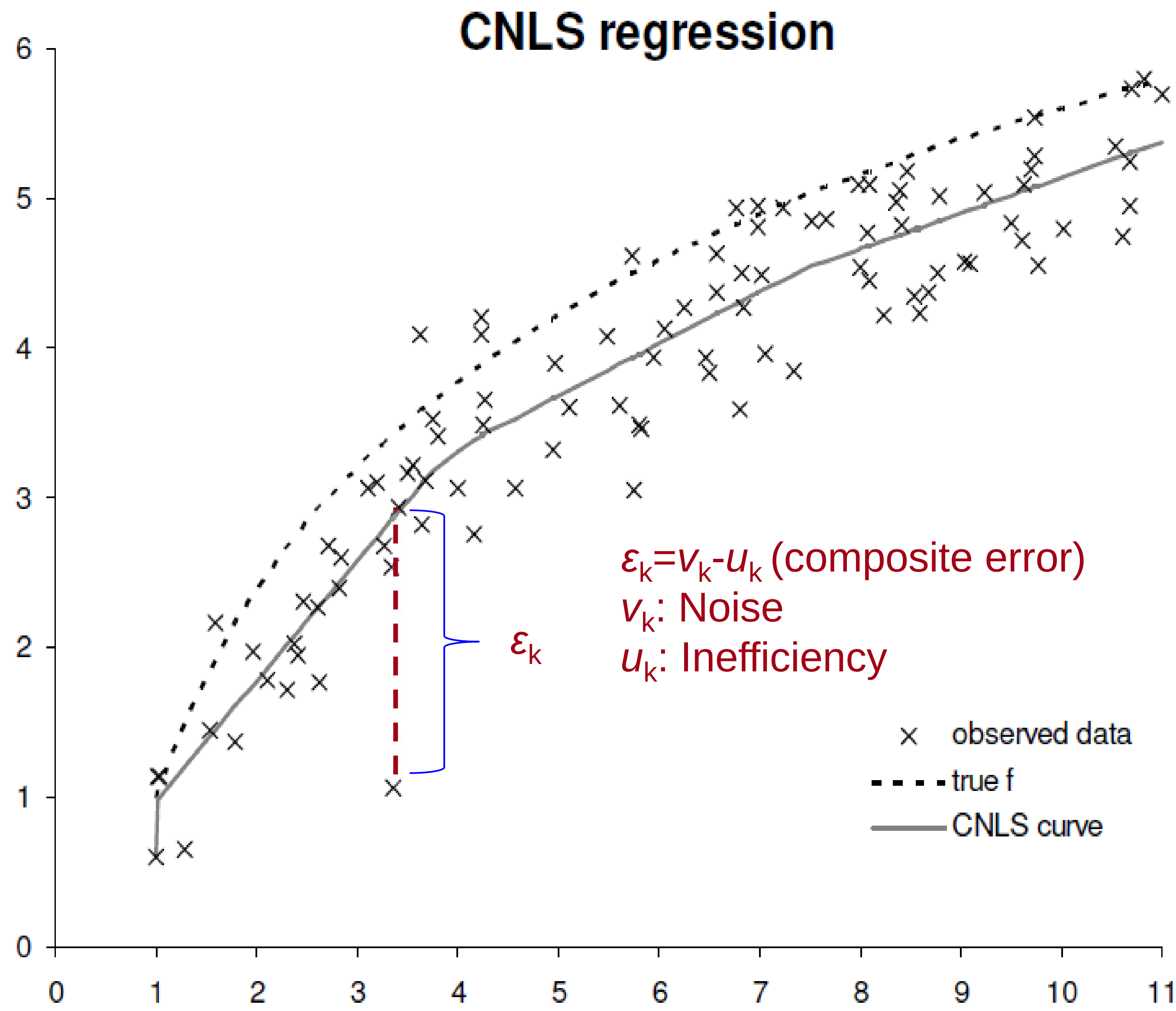
- Method of Moments is used to estimate
  - standard deviation parameter σ_u of the inefficiency distribution
  - standard deviation of the error term σ_v

- Estimation of the inefficiency term
  - Jondrow et al. (1982) have shown that the conditional distribution of inefficiency u_i given ε_i is a zero-truncated normal distribution with
    - Mean $-\varepsilon_i\sigma_u^2/(\sigma_u^2+\sigma_v^2)$; Variance $\sigma_u^2\sigma_v^2/(\sigma_u^2+\sigma_v^2)$
  - As a point estimator for u_i, one can use the conditional mean

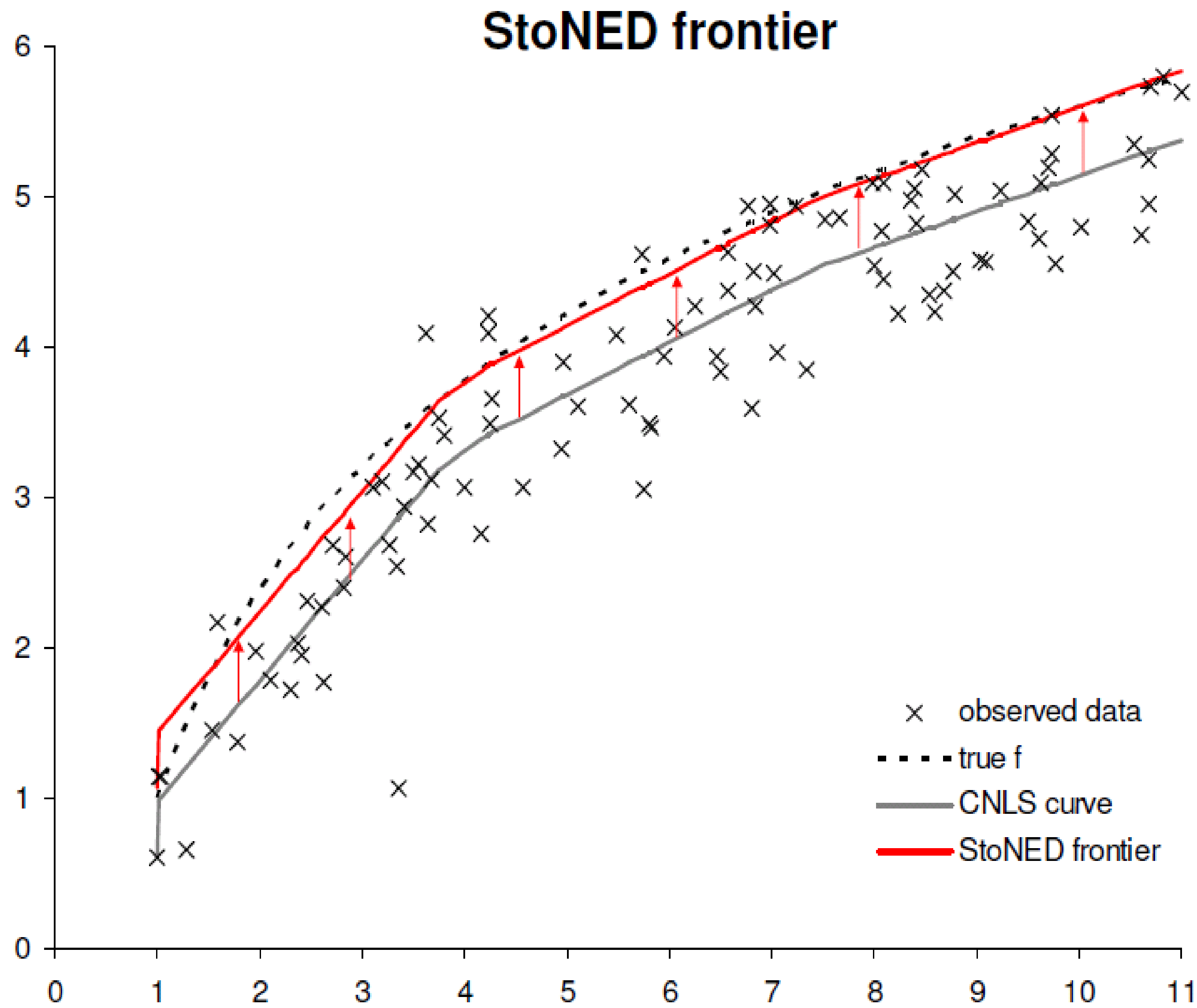
$$\hat{E}(u_i|\hat{\varepsilon}_i) = -\frac{\hat{\varepsilon}_i\hat{\sigma}_u^2}{\hat{\sigma}_u^2+\hat{\sigma}_v^2} + \frac{\hat{\sigma}_u^2\hat{\sigma}_v^2}{\hat{\sigma}_u^2+\hat{\sigma}_v^2}\left[\frac{\phi(\hat{\varepsilon}_i/\hat{\sigma}_v^2)}{1-\Phi(\hat{\varepsilon}_i/\hat{\sigma}_v^2)}\right]$$

  - ϕ the standard normal density function, and U is
  - Φ the standard normal cumulative distribution function
  - $\hat{\varepsilon}_i = \hat{\upsilon}_i - \hat{\sigma}_u\sqrt{2/\pi}$

Directional Marginal Productivity



Directional Marginal Productivity



Sign-Constraint CNLS

- CNLS addresses the limitations of SFA and DEA in estimating MP.
- Kuosmanen and Johnson (2010) demonstrated that inefficiency estimated by the sign-constrained CNLS is equivalent to that estimated by the additive single-output output-oriented DEA.

$$\varepsilon_k^{CNLS} = \text{argmin } \sum_k \varepsilon_k^2$$
$$\text{s.t. } \varepsilon_k \le 0, \forall k$$
$$Y_k = \alpha_k + \sum_i \beta_{ik} X_{ik} + \varepsilon_k, \forall k$$
$$\alpha_k + \sum_i \beta_{ik} X_{ik} \le \alpha_h + \sum_i \beta_{ih} X_{ik}, \forall k, \forall h$$
$$\beta_{ik} \ge 0, \forall i, k$$

Proposition 1: For all real-valued data, the MP estimated by sign-constrained convex nonparametric least-squares model with objective function $M \sum_k \varepsilon_k^2 + \sum_{i,k} \beta_{ik}$ is equivalent to the MP estimated by DEA model, that is, $\beta_{ik}^{+DEA} = \beta_{ik}^{+CNLS}$; the similar result can be applied to $\beta_{ik}^{-DEA} = \beta_{ik}^{-CNLS}$.

- It justifies that the value $\beta_{i^*j^*r}^{+DEA}$ proposed by P&F (2010) is the MP value β_{ik}^{+CNLS} of the regression-based CNLS approach.

Directional Marginal Productivity

Directional Distance Function (DDF)

- DDF estimates efficiency by expanding outputs and reducing inputs at the same time (Chambers et al., 1996; Chambers et al., 1998).
- Let $g = (g^X, g^Y)$ be the directional vector for inputs and outputs. We define the DDF as $D_o(x, y; g^X, g^Y)$, where η is the efficiency estimate.
  - If $\eta = 0$, then a firm r is efficient; otherwise the inefficient case.

$$
\begin{aligned}
D_o(x, y; g^X, g^Y) = &\text{ Max } \eta \\
\text{s.t. } & \textstyle\sum_k \lambda_k X_{i^*k} \le X_{ir} - \eta g^X,\ \forall i \\
& \textstyle\sum_k \lambda_k Y_{j^*k} \ge Y_{jr} + \eta g^Y,\ \forall j \\
& \textstyle\sum_k \lambda_k = 1 \\
& \lambda_k \ge 0,\ \eta \text{ is free}
\end{aligned}
$$

$\Rightarrow$

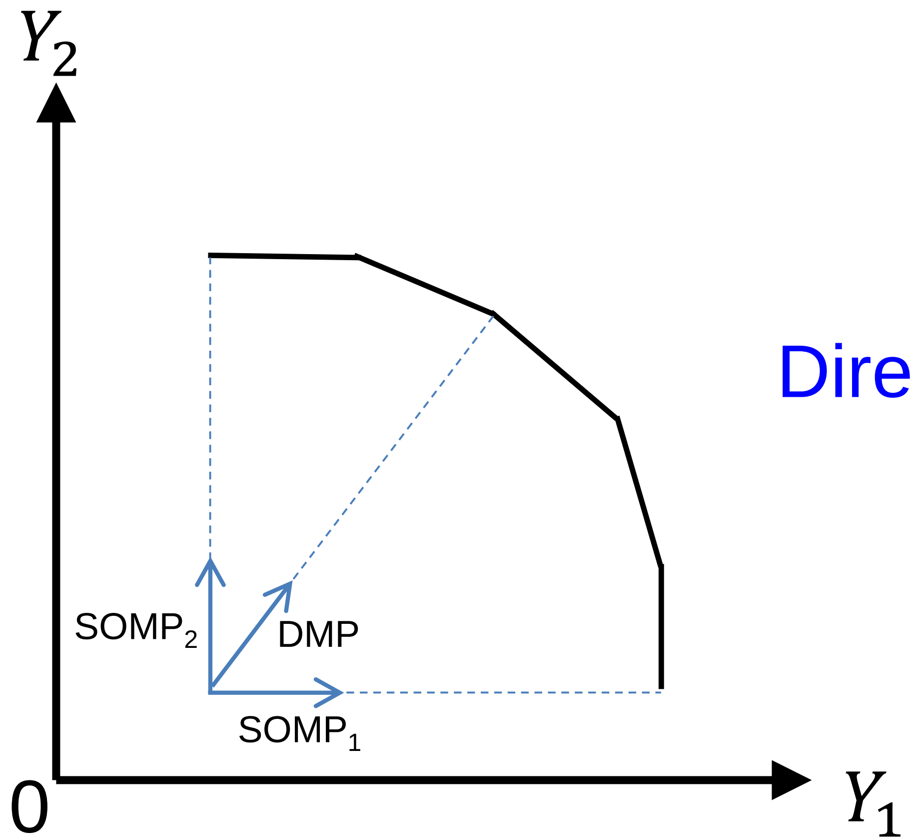
$$
\begin{aligned}
Y_{j^*r}(X_{i^*r}) = &\text{ Max } Y_{j^*r} + \eta g^{Y_{j^*}} \\
\text{s.t. } & \textstyle\sum_k \lambda_k X_{i^*k} \le X_{i^*r} - \eta g^{X_{i^*}} \\
& \textstyle\sum_k \lambda_k X_{ik} \le X_{ir},\ \forall i \neq i^* \\
& \textstyle\sum_k \lambda_k Y_{j^*k} \ge Y_{j^*r} + \eta g^{Y_{j^*}} \\
& \textstyle\sum_k \lambda_k Y_{jk} \ge Y_{jr},\ \forall j \neq j^* \\
& \textstyle\sum_k \lambda_k = 1 \\
& \lambda_k \ge 0,\ \eta \text{ is free}
\end{aligned}
$$

Lemma 1: Given $\left(g^{X_{i^*}}, g^{Y_{j^*}}\right) = (0,1)$ if firm r is on the efficient frontier, then model is equivalent to P&F's (2010) model representing the "maximum absolute level" of output.

Proposition 2: The single-output MP estimation by DEA, sign-constrained CNLS, or DDF with directional vector $\left(g^{X_{i^*}}, g^{Y_{j^*}}\right) = (0,1)$ will show a consistent result.

Directional Marginal Productivity

- How to develop the multi-output MP?
  - i.e., the multi-output MP shows the increases of multiple outputs when increasing one extra unit of input.
- How to present a trade-off among multiple outputs?



Direction!

Directional Marginal Productivity

Directional MP (DMP)

- DDF technique makes one-to-many mapping of MP possible. Let J^* be the output set whose marginal productivity will be investigated. Based on Prop.2, we estimate DMP by following model given the direction $(g^{X_{i^*}}, g^{Y_{j_1}}, g^{Y_{j_2}}, \ldots, g^{Y_{j_{|J^*|}}})$, where $g^{X_{i^*}} = 0$. Let $\sum_{j\in J^*} g^{Y_j} = 1$ for unit simplex (Färe et al., 2013).

$$
\begin{aligned}
&\text{Max } \eta\left(\textstyle\sum_{j\in J^*} g^{Y_j}\right) = \text{Max } \eta \\
&\text{s.t. } \textstyle\sum_k \lambda_k X_{i^*k} \le X_{i^*r} \\
&\quad \textstyle\sum_k \lambda_k X_{ik} \le X_{ir}, \forall i \ne i^* \\
&\quad \textstyle\sum_k \lambda_k Y_{jk} \ge Y_{jr} + \eta g^{Y_j}, \forall j \in J^* \\
&\quad \textstyle\sum_k \lambda_k Y_{jk} \ge Y_{jr}, \forall j \in J \setminus J^* \\
&\quad \textstyle\sum_k \lambda_k = 1 \\
&\quad \lambda_k \ge 0, \eta \text{ is free}
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
&\text{Min } v_{i^*} \\
&\text{s.t. } \textstyle\sum_i v_i X_{ir} - \sum_j u_j Y_{jr} + u_0 = 0 \\
&\quad \textstyle\sum_i v_i X_{ik} - \sum_j u_j Y_{jk} + u_0 \ge 0, \forall k \\
&\quad \textstyle\sum_{j\in J^*} u_j g^{Y_j} = 1 \\
&\quad v_i, u_j \ge 0, u_0 \text{ is free}
\end{aligned}
$$

- Note that $(g^{Y_{j_1}}, g^{Y_{j_2}}, \ldots, g^{Y_{j_{|J^*|}}})$ can be regarded as the "weightings" between investigated outputs. The larger the weight, the closer the direction will be to the output with the higher weight.

PCLab

Directional MP

- Eliminate the unit of each factor

$$\text{Min } \frac{v_{i^*}}{X_{i^*}^{Max}} = \alpha$$

$$\text{s.t. } \sum_i v_i \frac{X_{ir}}{X_i^{Max}} - \sum_j u_j \frac{Y_{jr}}{Y_j^{Max}} + u_0 = 0$$

$$\sum_i v_i \frac{X_{ik}}{X_i^{Max}} - \sum_j u_j \frac{Y_{jk}}{Y_j^{Max}} + u_0 \geq 0, \forall k$$

$$\sum_{j \in J^*} u_j g^{Y_j} = 1$$

$$v_i, u_j \geq 0, u_0 \text{ is free}$$

The reason for introducing unit simplex and eliminating the measurement units of inputs and outputs is to normalize the weight which presents a tradeoff among outputs.

- Therefore, increasing one extra unit of X_{i^*} of firm r, means that the vector of the multi-product MP with respect to output Y_{j^*} is $\frac{\partial Y_{jr}}{\partial X_{i^*r}} = \alpha \times \left(g^{Y_j} Y_j^{Max}\right), \forall j \in J^*$.

Directional Marginal Productivity (DMP)

- How the change of single input X_{i^*} affects the multiple outputs
  - Let set $J^* \subset J$ be the outputs set investigated. Given the direction vector $(g^{X_{i^*}}, g^{Y_j})$ as parameters, where $g^{X_{i^*}} = 0$ and $\sum_{j \in J^*} g^{Y_j} = 1$ for unit simplex (Färe et al., 2013). Let $X_i^{Max} = \max\{X_{ik}\}$ and $Y_j^{Max} = \max\{Y_{jk}\}$

$$\alpha = \text{Min } \frac{v_{i^*}}{X_{i^*}^{Max}}$$

$$\text{s.t. } \sum_i v_i \frac{X_{ir}}{X_i^{Max}} - \sum_j u_j \frac{Y_{jr}}{Y_j^{Max}} + u_0 = 0$$

$$\sum_i v_i \frac{X_{ik}}{X_i^{Max}} - \sum_j u_j \frac{Y_{jk}}{Y_j^{Max}} + u_0 \geq 0, \forall k$$

$$\sum_{j \in J^*} u_j g^{Y_j} = 1$$

$$v_i, u_j \geq 0, u_0 \text{ is free}$$

The reason for introducing unit simplex and eliminating the measurement units of inputs and outputs is to normalize the weight which presents a tradeoff among outputs.

  - increasing one extra unit of X_{i^*} of firm r, means that the vector of the DMP with respect to output Y_j is $\frac{\partial Y_{jr}}{\partial X_{i^*r}} = \alpha \times \left(g^{Y_j} Y_j^{Max}\right), \forall j \in J^*$.

Directional Marginal Productivity

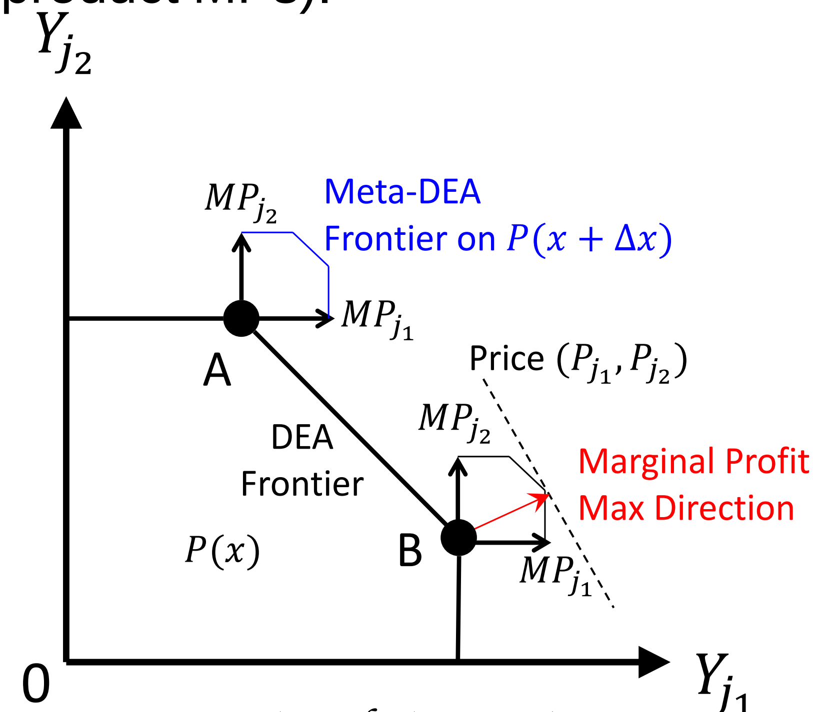
Proposition 3: If the direction for MP estimation used in proposed model projects to the portion of free disposability with respect to inputs, then the DMP estimate will be equal to 0.

Proposition 4: The marginal productivity estimated by the proposed model with the objective function $\mathrm{Max}\ \frac{v_{i^*}}{X_{i^*}^{Max}} = \alpha$ is equivalent to the marginal productivity estimated, given a negative direction.

Directional Marginal Productivity

Meta-DEA

- an approach to find a direction for an efficient firm to move towards its allocatively efficient benchmark based on maximization of the firm's marginal profits.
- We can generate the DMPs manually, given random-picked directions, and then calculate the allocative efficiency with respect to the meta-DEA frontier based on these discrete directions/observations (i.e., vectors of multi-product MPs).

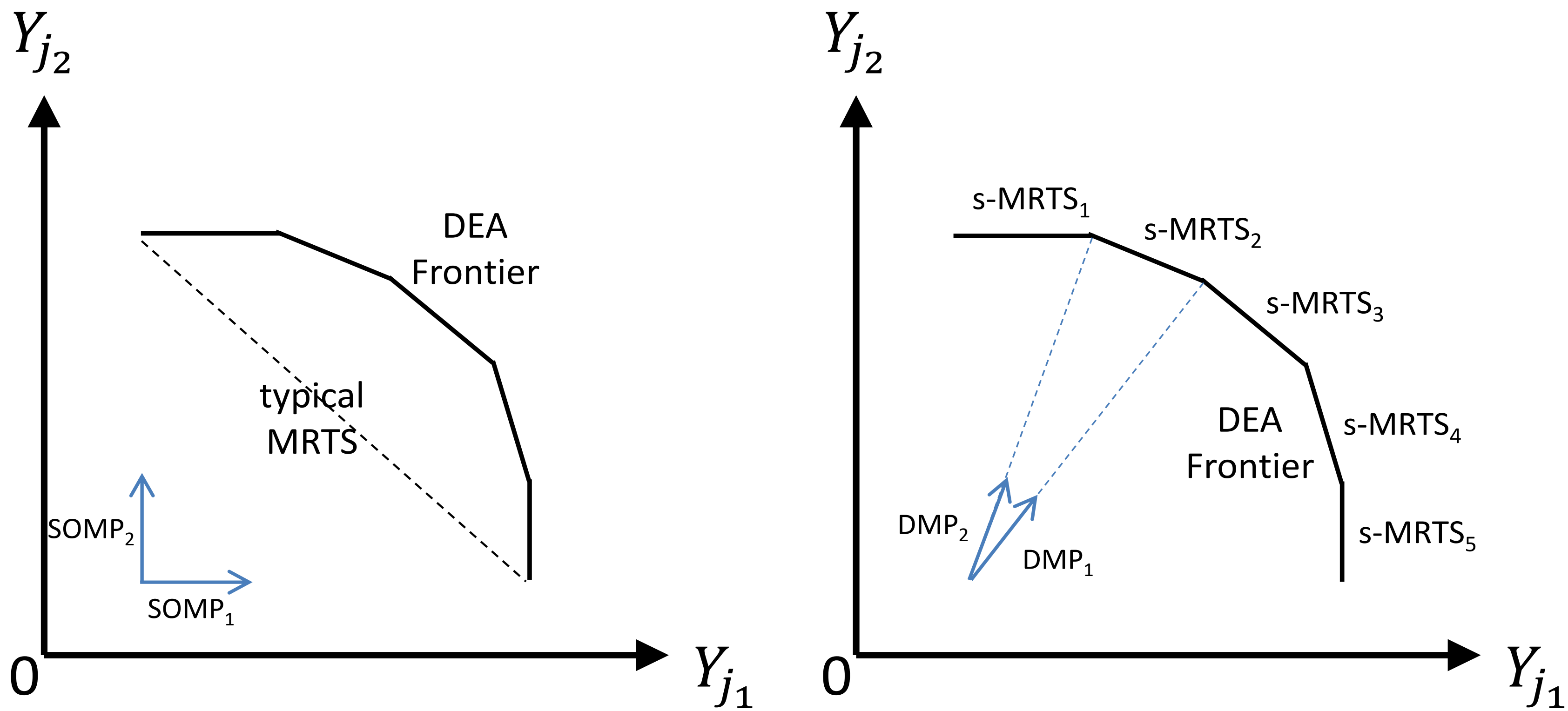


Marginal rate of technical substitution (MRTS)

- Arbitrary two outputs Y_{j_1} and Y_{j_2} , one-sided $MRTS^{+} = \frac{-MP_{j_1}}{MP_{j_2}} = \frac{\alpha_1 Y_{j_1}^{Max}}{\alpha_2 Y_{j_2}^{Max}}$, where $j_1, j_2 \in J^*$, α_1 is calculated given the direction $(g^{Y_{j_1}}, g^{Y_{j_2}}) = (1,0)$, and α_2 is calculated by $(g^{Y_{j_1}}, g^{Y_{j_2}}) = (0,1)$, which indicates a single-output MP respectively.

Definition 1: Segmented marginal rate of technical substitution (s-MRTS) can be calculated by investigating two specific outputs and defined as *s-MRTS*$^{+}$= $\frac{\Delta MP_{j_1}}{\Delta MP_{j_2}} = \frac{(\alpha_1 g_1^{Y_{j_1}} Y_{j_1}^{Max} - \alpha_2 g_2^{Y_{j_1}} Y_{j_1}^{Max})}{(\alpha_1 g_1^{Y_{j_2}} Y_{j_2}^{Max} - \alpha_2 g_2^{Y_{j_2}} Y_{j_2}^{Max})}$, where $DMP_1^{+} = (\alpha_1 g_1^{Y_{j_1}} Y_{j_1}^{Max}, \alpha_1 g_1^{Y_{j_2}} Y_{j_2}^{Max})$ and $DMP_2^{+} = (\alpha_2 g_2^{Y_{j_1}} Y_{j_1}^{Max}, \alpha_2 g_2^{Y_{j_2}} Y_{j_2}^{Max})$ are the two vectors of the DMPs.

Directional Marginal Productivity



(a) MRTS (b) segmented MRTS

Figure MRTS and segmented MRTS: (a) two single-output MPs (SOMP) are used to calculate MRTS as dash line; (b) two DMPs (DMP) are used to calculated s-MRTS$_2$ as one piece-wise line segment on frontier

Directional Marginal Productivity

DMP

- Return to the example in Podinovski and Førsund (2010), which includes one input, two outputs, and three observations (firms A, B, and C in following Table)

Unit	Input (X_1)	Output 1 (Y_1)	Output 2 (Y_2)
A	2	1	200
B	4	2	300
C	1	4	100

- Formulation of firm A

$$\text{Min } \frac{v_1}{4} = \alpha$$

$$\text{s.t. } v_1 \frac{2}{4} - u_1 \frac{1}{4} - u_2 \frac{200}{300} + u_0 = 0$$

$$v_1 \frac{4}{4} - u_1 \frac{2}{4} - u_2 \frac{300}{300} + u_0 \geq 0$$

$$v_1 \frac{1}{4} - u_1 \frac{4}{4} - u_2 \frac{100}{300} + u_0 \geq 0$$

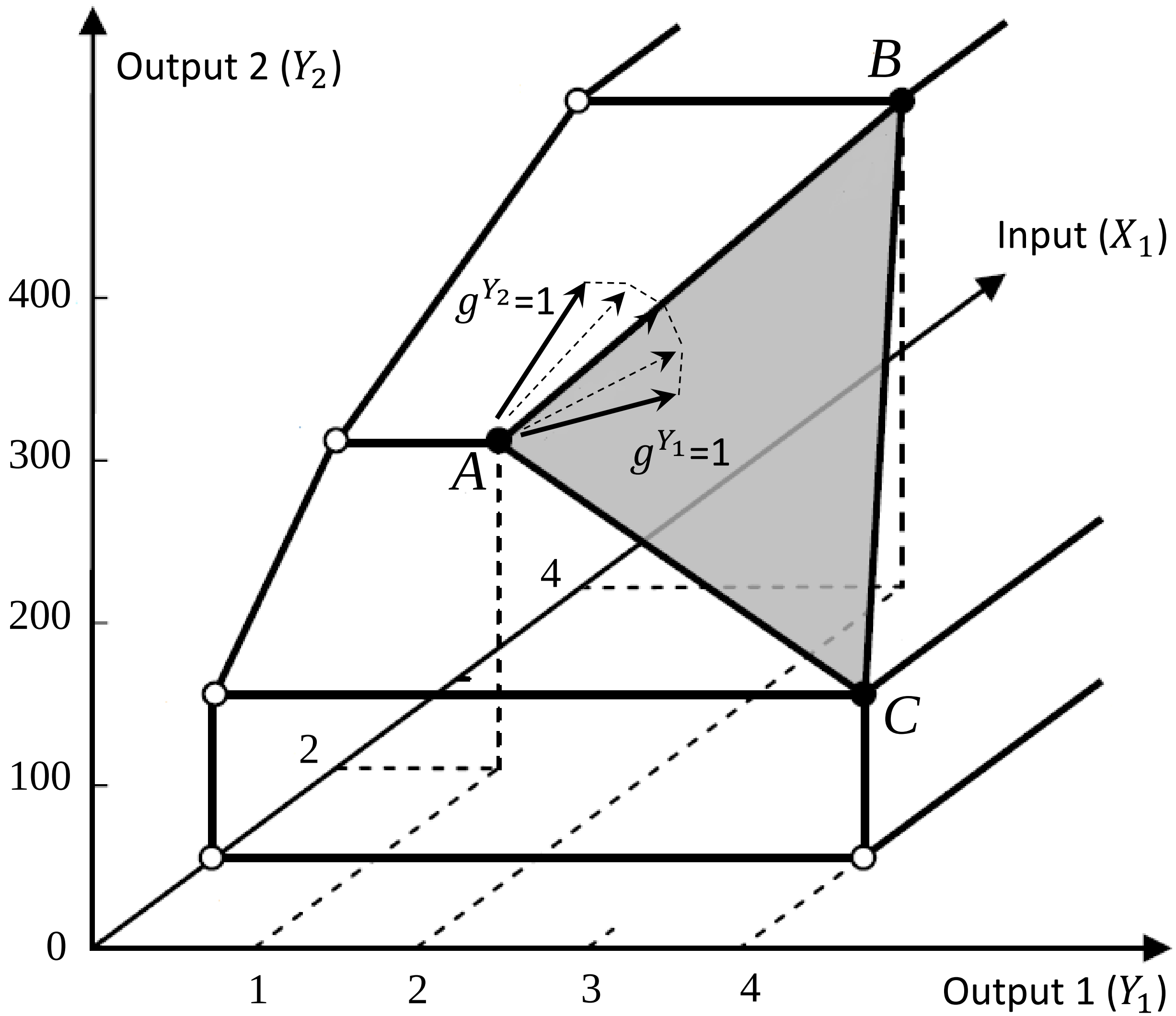
$$u_1 g^{Y_1} + u_2 g^{Y_2} = 1$$

$v_1, u_1, u_2 \geq 0$, u_0 is free

$$\frac{\partial (Y_{1A}, Y_{2A})}{\partial X_{1A}} = \alpha \times (4g^{Y_1}, 300g^{Y_2})$$

Directional Marginal Productivity

- DMP in firm A (Desirable Output Substitution with One Extra Unit of Input)



Directional Marginal Productivity

- DMP and meta-DEA of Y_1 and Y_2 in firm A

Case No.	Direction (normalized)		Objective Function	DMP	s-MRTS	Meta-DEA
	g^{Y_1}	g^{Y_2}	α	$\frac{\partial(Y_{1A}, Y_{2A})}{\partial X_{1A}}$		$\frac{P_1}{P_2}$
Case 1	1	0	1	(4, 0)		[14.3198, ∞)
Case 2	0.9	0.1	0.70	(2.53, 21.05)	-0.07	[14.26X, 14.3198)
Case 3	0.8	0.2	0.54	(1.73, 32.43)	-0.07	[14.26X, 14.26X)
Case 4	0.7	0.3	0.44	(1.23, 39.56)	-0.07	[14.26X, 14.26X)
Case 5	0.6	0.4	0.37	(0.89, 44.44)	-0.07	[14.24, 14.26X)
Case 6	0.5	0.5	0.32	(0.64, 48.00)	-0.07	[10, 14.24)
Case 7	0.4	0.6	0.28	(0.44, 50.00)	-0.098	(0, 10)
Case 8	0.3	0.7	0.24	(0.29, 50.00)	N/A	
Case 9	0.2	0.8	0.21	(0.17, 50.00)	N/A	
Case 10	0.1	0.9	0.19	(0.07, 50.00)	N/A	
Case 11	0	1	0.167	(0.00, 50.00)	N/A	

Directional Marginal Productivity

- Summary
  - Directional marginal productivity promotes of Podinovski and Førsund's (2010) work.
  - push the typical "ex-post" DEA study towards a "ex-ante" DEA.
  - A well-defined connection between DEA, sign-constrained CNLS, and DDF.
  - A measure of s-MRTS to compensate the typical MRTS measure via a segmentation technique and calculation of each output's marginal difference

- Issues
  - No capture synergistic effects
    - Noting that the estimation of the increase in output is conservative if two or more inputs are expanded simultaneously.
  - Support capacity adjustment
    - but moving along the efficient frontier too far may be out of production possibility set.

Directional Marginal Productivity

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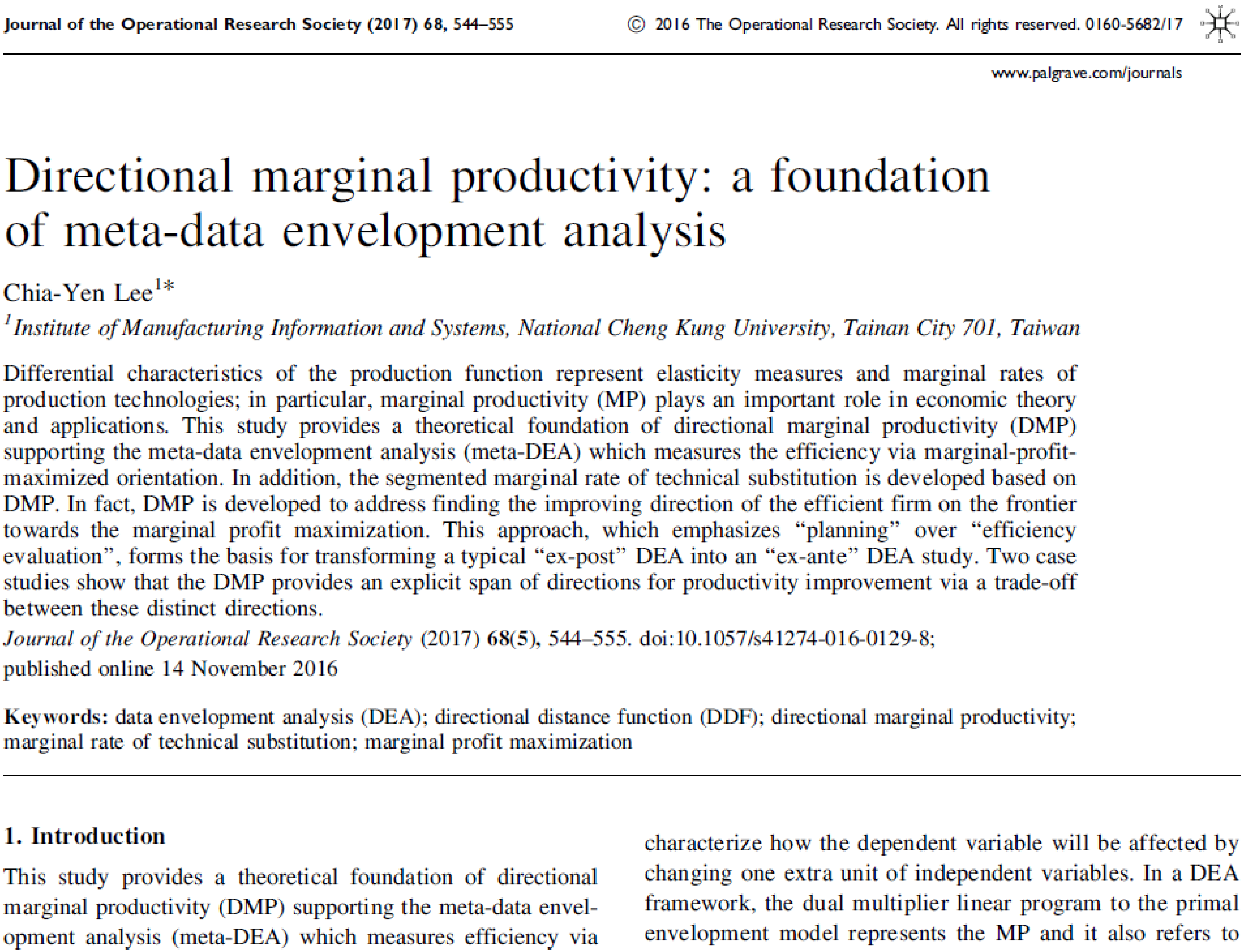

Journal of the Operational Research Society (2017) 68, 544–555

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Directional marginal productivity: a foundation of meta-data envelopment analysis

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Differential characteristics of the production function represent elasticity measures and marginal rates of production technologies; in particular, marginal productivity (MP) plays an important role in economic theory and applications. This study provides a theoretical foundation of directional marginal productivity (DMP) supporting the meta-data envelopment analysis (meta-DEA) which measures the efficiency via marginal-profit-maximized orientation. In addition, the segmented marginal rate of technical substitution is developed based on DMP. In fact, DMP is developed to address finding the improving direction of the efficient firm on the frontier towards the marginal profit maximization. This approach, which emphasizes "planning" over "efficiency evaluation", forms the basis for transforming a typical "ex-post" DEA into an "ex-ante" DEA study. Two case studies show that the DMP provides an explicit span of directions for productivity improvement via a trade-off between these distinct directions.

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Keywords: data envelopment analysis (DEA); directional distance function (DDF); directional marginal productivity; marginal rate of technical substitution; marginal profit maximization

1. Introduction

This study provides a theoretical foundation of directional marginal productivity (DMP) supporting the meta-data envelopment analysis (meta-DEA) which measures efficiency via

characterize how the dependent variable will be affected by changing one extra unit of independent variables. In a DEA framework, the dual multiplier linear program to the primal envelopment model represents the MP and it also refers to

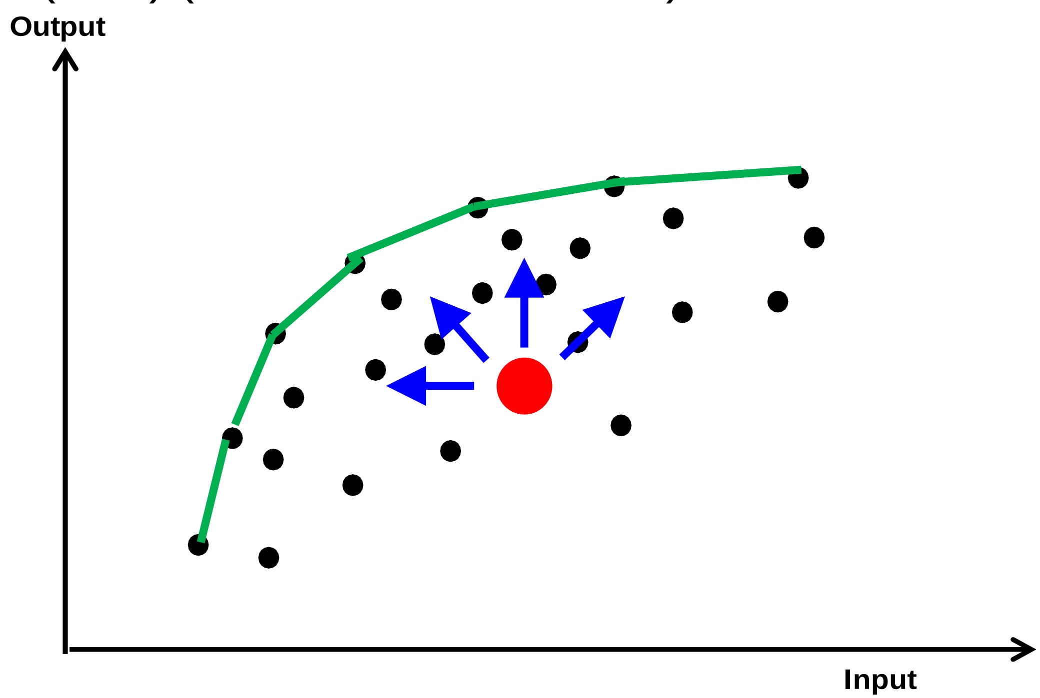
Meta-DEA

Lee, Chia-Yen and A. L. Johnson, 2012. Two-dimensional Efficiency Decomposition to Measure the Demand Effect in Productivity Analysis. European Journal of Operational Research, 216 (3), 584–593.

Meta-DEA (Lee, 2014)

Orientation?

- Debteu (1951) and Farrell (1957) developed radial measure.
- a Farrell efficient may not be Koopmans efficient (Koopmans, 1951).
  - the Russell measure (Färe & Lovell, 1978), the additive DEA model (Charnes et al., 1985), and the slacks-based measure (Tone, 2001)
- contract input level and expand output level simultaneously
  - the hyperbolic measure (Färe et al., 1985) and the directional distance function (DFF) (Chambers et al., 1996)



- Kuosmanen and Podinovski (2009) introduce the weak disposability property which forms a convex technology with undesirable outputs.
- MP for multiple outputs → direction (g^{Y_j}, g^{B_q})

$$\text{Min } v_{i^*}$$

$$\text{s.t. } \sum_i v_i \frac{X_{ir}}{X_i^{Max}} - \sum_j u_j \frac{Y_{jr}}{Y_j^{Max}} + \sum_q w_q \frac{B_{qr}}{B_q^{Max}} + u_0 = 0$$

$$\sum_i v_i \frac{X_{ik}}{X_i^{Max}} - \sum_j u_j \frac{Y_{jk}}{Y_j^{Max}} + \sum_q w_q \frac{B_{qr}}{B_q^{Max}} + u_0 \geq 0, \forall k$$

$$\sum_i v_i \frac{X_{ik}}{X_i^{Max}} + u_0 \geq 0, \forall k$$

$$\frac{\partial(Y_{jr}, B_{qr})}{\partial X_{i^*r}} = v_{i^*}\left(g^{Y_j} Y_j^{Max}, -g^{B_q} B_q^{Max}\right) / X_{i^*}^{Max}$$

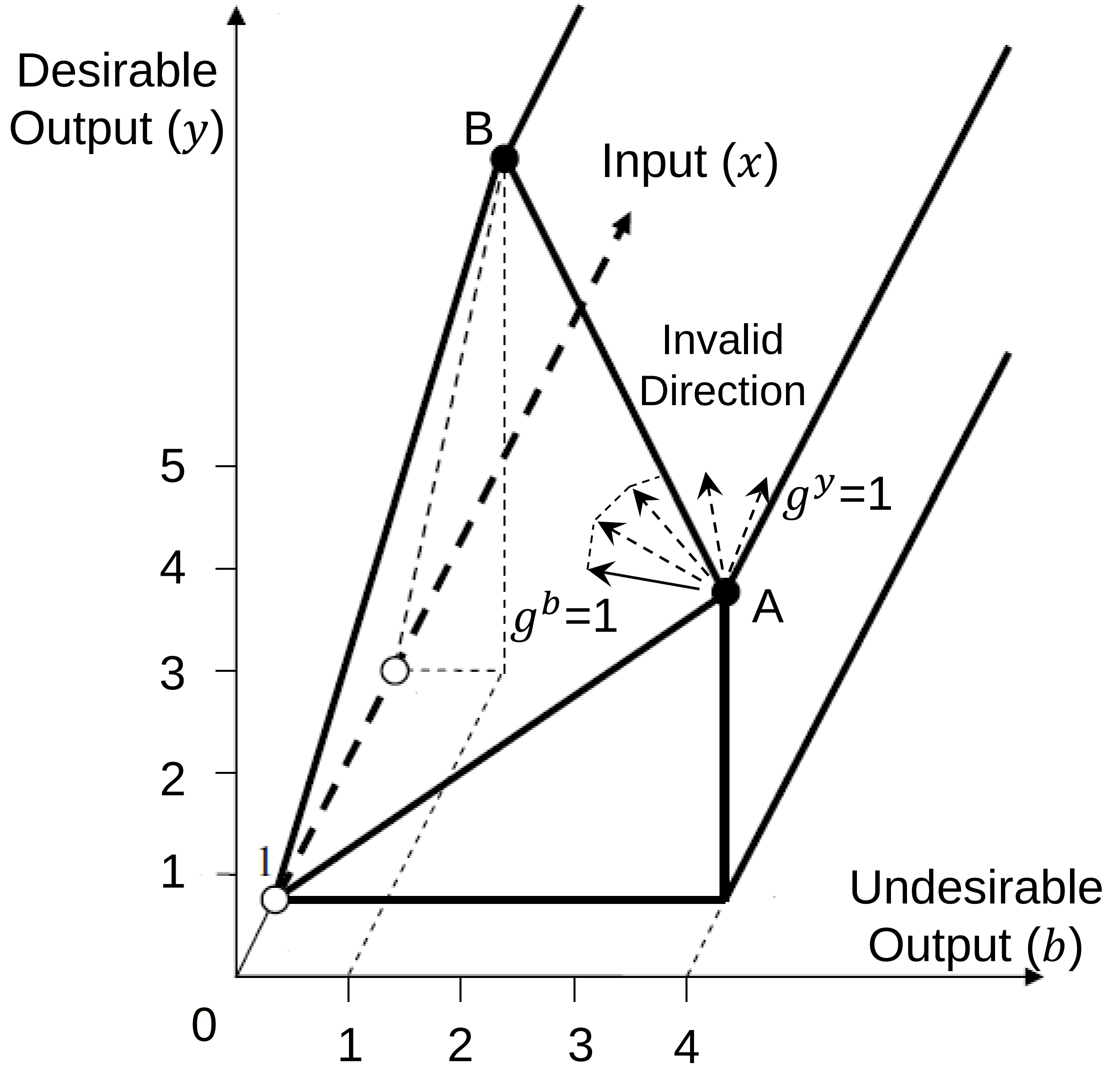
Directional Marginal Productivity

$$\sum_{j \in J^*} u_j g^{Y_j} + \sum_{q \in Q^*} w_q g^{B_q} = 1$$

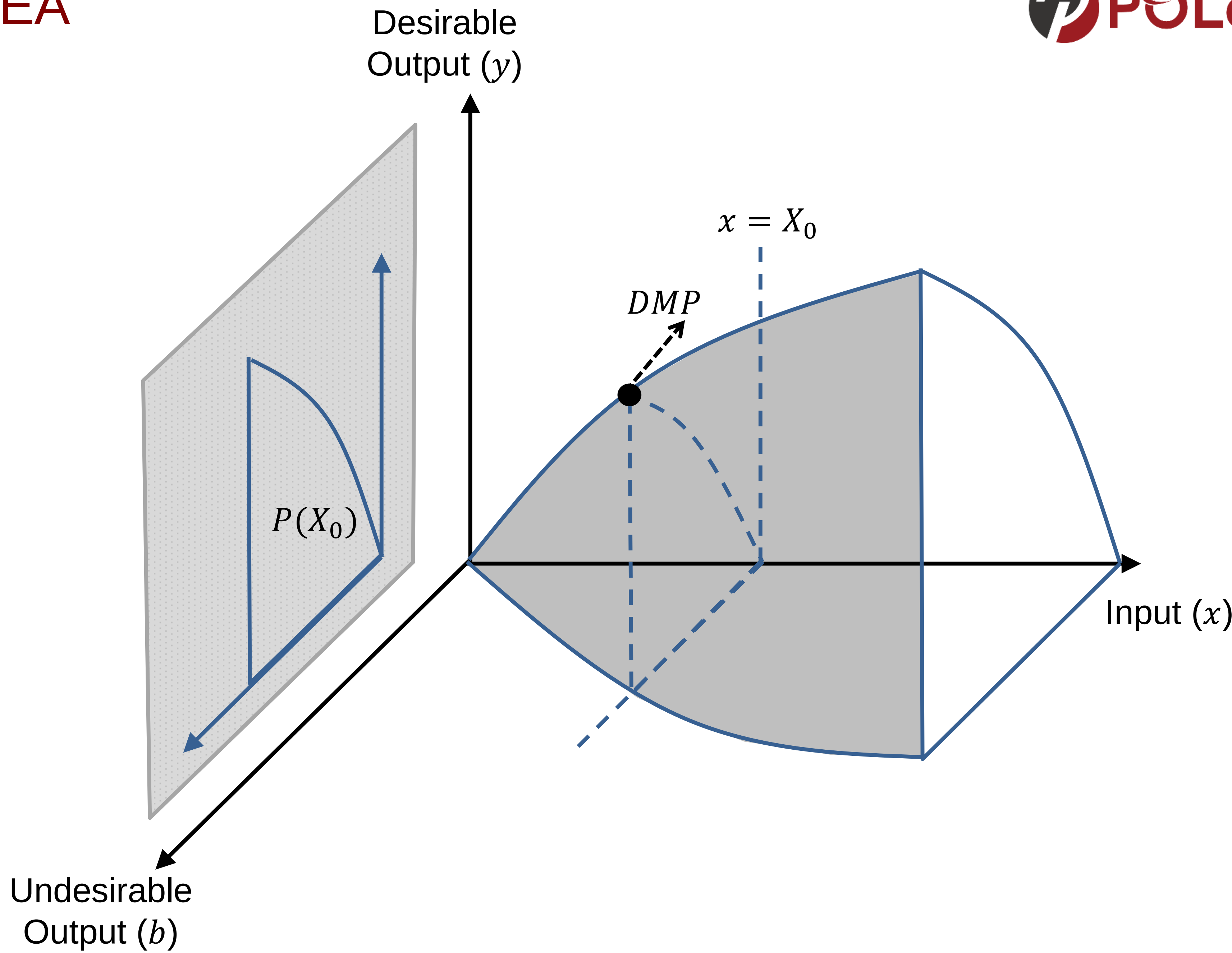
$v_i, u_j \geq 0$, w_q, u_0 are free

Note: $\sum_{j \in J^*} g^{Y_j} + \sum_{q \in Q^*} g^{B_q} = 1$ for unit simplex (Färe et al., 2013)

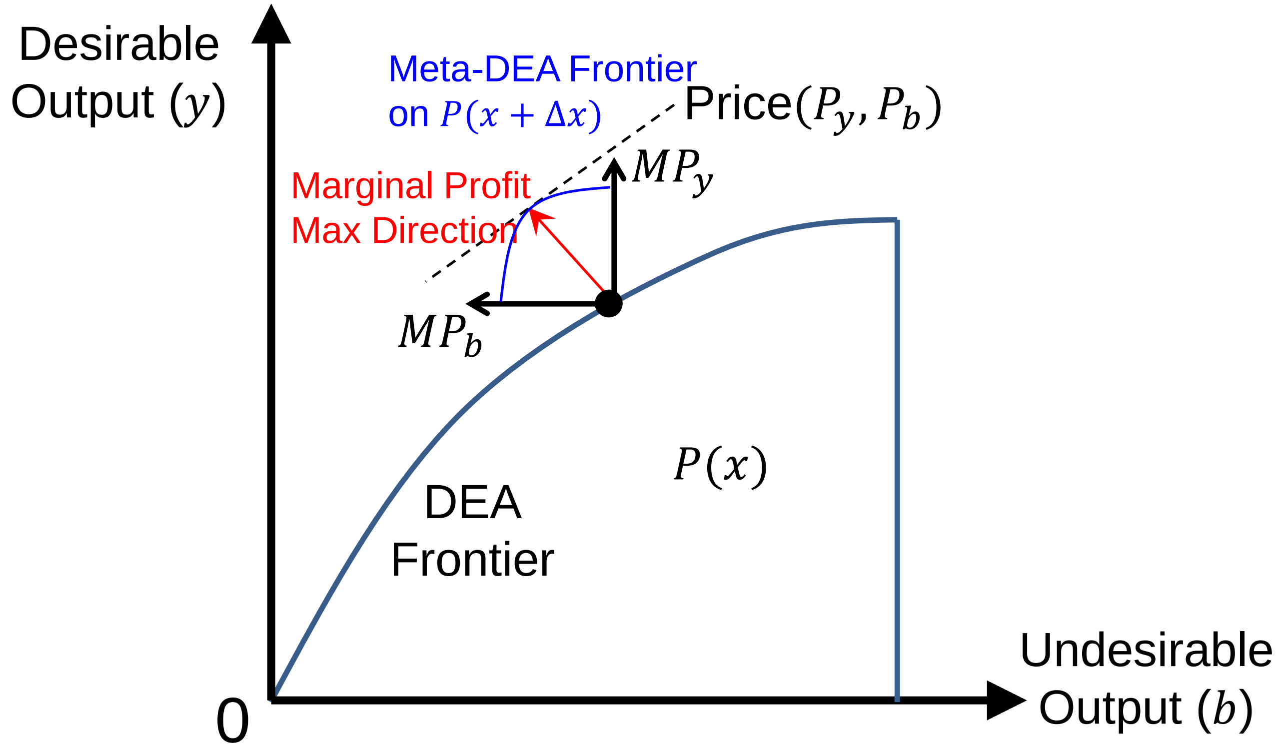
Illustration of DMP of Y_1 and B_1 in firm A



Meta-DEA



Meta-DEA for Marginal Profit Maximization



$$\left(g^*_{y_j}, g^*_{b_q}\right) = \underset{g^{\psi}_{y_j}, g^{\psi}_{b_q}}{\arg\max} \left\{ \left(\sum_{j \in J^*} P_{y_j} g^{\psi}_{y_j} Y^{Max}_j + \sum_{q \in Q^*} P_{b_q} g^{\psi}_{b_q} B^{Max}_q \right) v^{\psi}_{i^*} / X^{Max}_{i^*} - P_{x_{i^*}} \,\middle|\, \text{model} \quad \text{given } \left(P_{x_i}, P_{y_j}, P_{b_q}\right),\ \forall \psi \right\}$$

Meta-DEA (Lee, 2014)

- Efficiency estimation for desirable and undesirable outputs

$$\begin{aligned}
&\text{Max } \eta \\
&\text{s.t.} \quad \sum_k (\lambda_k + \mu_k) X_{ik} \le X_{ir} + \eta g_{x_i}, \quad \forall i \in I \\
&\sum_k \lambda_k Y_{jk} \ge Y_{jr} + \eta g_{y_j}, \quad \forall j \in J \\
&\sum_k \lambda_k B_{qk} = B_{qr} - \eta g_{b_q}, \quad \forall q \in Q \\
&\sum_k (\lambda_k + \mu_k) = 1 \\
&\lambda_k, \mu_k \geqslant 0, \quad \eta \text{ is free}
\end{aligned}$$

- Comparison of different direction vectors for 5 eff measures

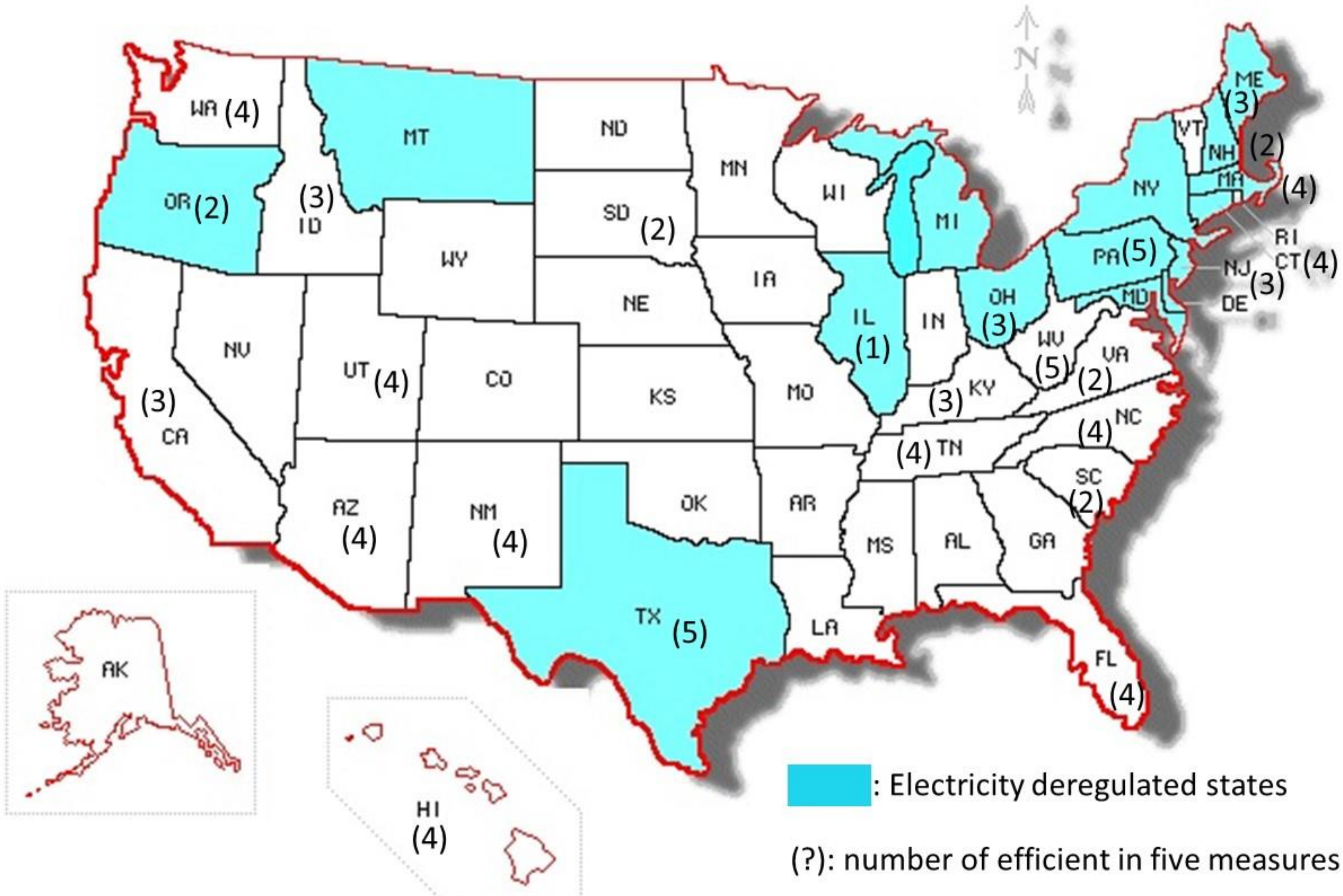
Study	Name	Proposed directional vector (g_x, g_y, g_b)	Price data
Färe et al. (2006)	DDF1	$(0, 1, 1)$	No
Chung et al. (1997)	DDF2	$(0, y, b)$	No
Färe et al. (2013)	DDF3	$\left(0, \frac{\eta^*_{y_j}}{\sum_{j\in J}\eta^*_{y_j} + \sum_{q\in Q}\eta^*_{b_q}}, \frac{\eta^*_{b_q}}{\sum_{j\in J}\eta^*_{y_j} + \sum_{q\in Q}\eta^*_{b_q}}\right)$	No
Zofio et al. (2013)	DDF4	$\frac{(x^* - x, y^* - y, b - b^*)}{(P_y y^* - P_b b^* - P_x x^*) - (P_y y - P_b b - P_x x)}$	Yes
This paper	DDF5	$\left(1, g^*_{y_j} Y_j^{Max} v_{i^*} / X_{i^*}^{Max}, g^*_{b_q} B_q^{Max} v_{i^*} / X_{i^*}^{Max}\right)$	Yes

Meta-DEA (Lee, 2014)

- Statistics for U.S. state-level coal power units operating in 2011.

Variable	Mean	Std. dev
Coal (tons)	19,477,882	20,161,963
Electricity (mehawatt hour)	36,113,125	35,089,755
CO_2 (tons)	40,591,530	38,931,017
SO_2 (tons)	101,895	128,948
NO_x (tons)	40,154	35,360
Coal price ($/ton)	46.0	20.5
Elect. price ($/mehawatt hour)	101.4	40.3
CO_2 price ($/ton)	18.1	0.0
SO_2 price ($/ton)	470.0	0.0
NO_x price ($/ton)	1056.3	0.0

- Electricity deregulated states and efficiency index.



- Correlation coefficient matrix of the five efficiency measures.

Correlation coefficient	DDF1	DDF2	DDF3	DDF4	DDF5
DDF1	1.000	0.628*	0.536*	0.052	0.294*
DDF2		1.000	0.895*	0.007	0.456*
DDF3			1.000	−0.190	0.446*
DDF4				1.000	−0.147
DDF5					1.000

* Pass significant testing at level 0.05.

- Summary
  - We observe that only Pennsylvania (PA), Texas (TX), and West Virginia (WV) are efficient in five measures; PA is deregulated and TX is partially deregulated.
  - Of the 15 electricity deregulated states, 10 are efficient according to at least one of the five efficiency measures. Generally, the deregulated states have more health and environmental benefits due to pollutant reductions and emissions trading (Fowlie, 2010). Thus, these benefits may lead to higher efficiency scores in the deregulated states.

Meta-DEA

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Decision Support

Meta-data envelopment analysis: Finding a direction towards marginal profit maximization

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A R T I C L E   I N F O

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Marginal profit maximization
Directional marginal productivity
Coal-fired power plant

A B S T R A C T

This paper discusses a new meta-DEA approach to solve the problem of choosing direction vectors when estimating the directional distance function. The proposed model emphasizes finding the "direction" for productivity improvement rather than estimating the "score" of efficiency; focusing on "planning" over "evaluation". In fact, the direction towards marginal profit maximization implies a step-by-step improvement and "wait-and-see" decision process, which is more consistent with the practical decision-making process. An empirical study of U.S. coal-fired power plants operating in 2011 validates the proposed model. The results show that the efficiency measure using the proposed direction is consistent with all other indices with the exception of the direction towards the profit-maximized benchmark. We conclude that the marginal profit maximization is a useful guide for determining direction in the directional distance function.

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Proactive DMP for Business Shutdown Decision

Lee, Chia-Yen, 2019. Proactive Marginal Productivity Analysis for Business Shutdown Decision by DEA. Journal of the Operational Research Society, 70(7), 1065-1078.

Business Shutdown Decision

Background and Motivation

- The decision to shut a business usually arises when the product's marginal revenue falls below the average variable cost since the firm cannot offset the fixed cost.
- Hhowever, this traditional business shutdown criterion (BSC) as defined by microeconomic theory and simple output-input financial ratios do not reflect all of the critical factors in performance evaluation (Chen and McGinnis, 2007), and thus may no longer apply to some types of industry; eg. the high-tech industry due to its capital-intensity.
- However, in practice, many business are profit-seeking but not always profit maximizing (Nelson, 2011), i.e., a variety of KPIs.
- a tradeoff between short-run aspect and long-run aspect generally confuses the economic analysis framework by sacrificing short-run profit for long-run profit maximization.
- This motivates our study, for surviving in a period of economic recession, the Taiwan's LED manufacturers in 2012 aimed to maintain market share at the expense of a rival by cutting prices or expanding capacity.

Business Shutdown Decision

Literature

- the short-run production decision depends on the price, average total cost (ATC), and average variable cost (AVC).
- In a competitive market, for profit maximization with price larger than ATC, this case generates a positive economic profit.
  - That is, in this case a firm continues producing at the quantity that equates marginal revenue and marginal cost (Lipsey, 1975).
- However, if price is lower than ATC but higher than AVC, then this case is for loss minimization and a firm can keep producing to obtain enough revenue to cover all variable cost plus partial fixed cost until marginal revenue equals marginal cost. Note that if all fixed costs are non-sunk, then a firm may shut down if the price is below ATC (Pindyck and Rubinfeld, 2001).
- Finally, if price is lower than AVC, a firm should shut the production line in the short run since the loss incurred is greater than fixed cost, and thereby the firm can incur a smaller loss by producing zero output and paying fixed cost (Perloff, 2014).

Business Shutdown Decision

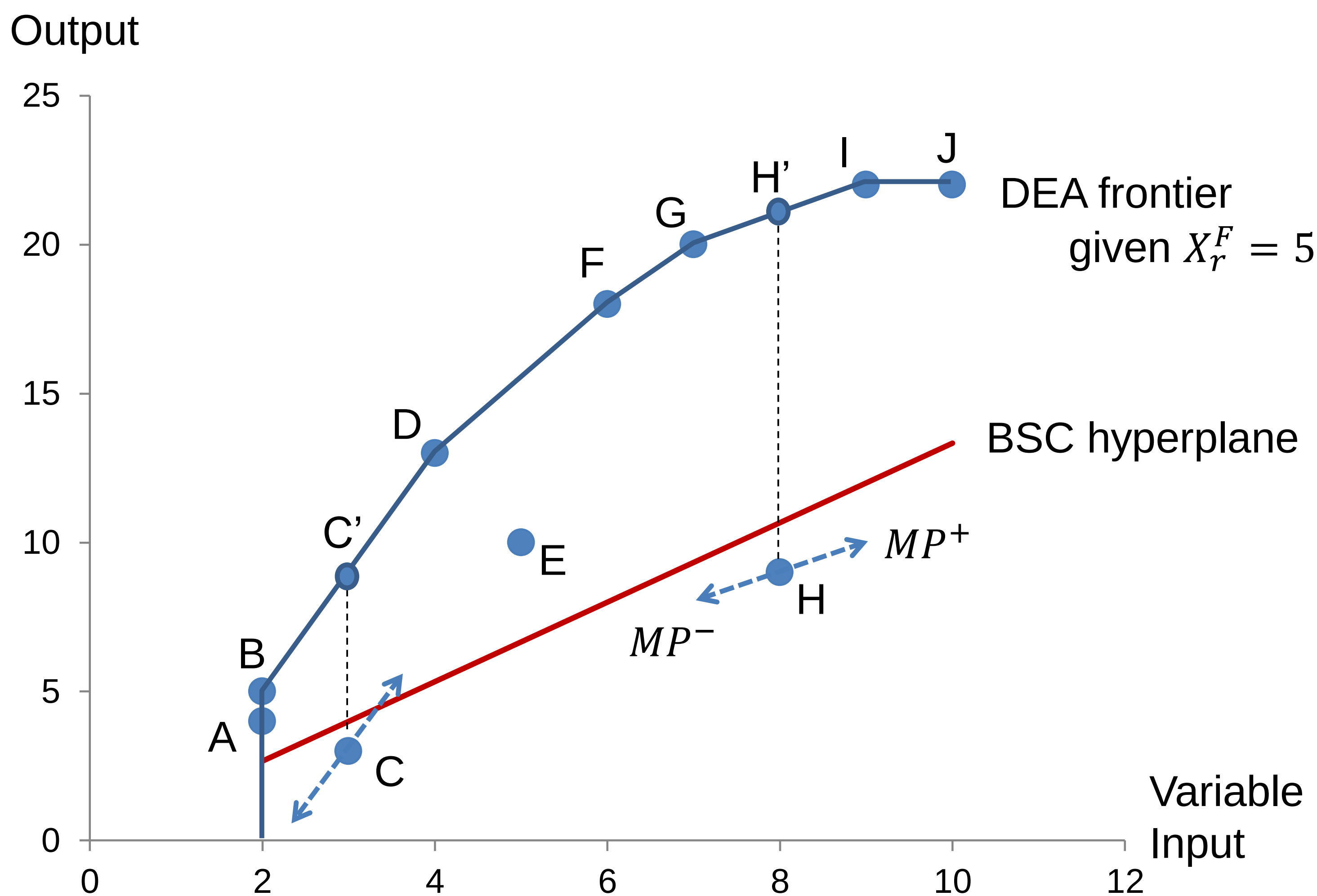
Research Aims

- This study proposes a business shutdown criterion (BSC) scheme that uses an iterative procedure embedded with three phases: level analysis, margin analysis, and budget and action, to solve the shutdown decision problem via proactive marginal productivity (MP).

- It improves understanding why firms continue losing money while staying in the market despite intense competition.

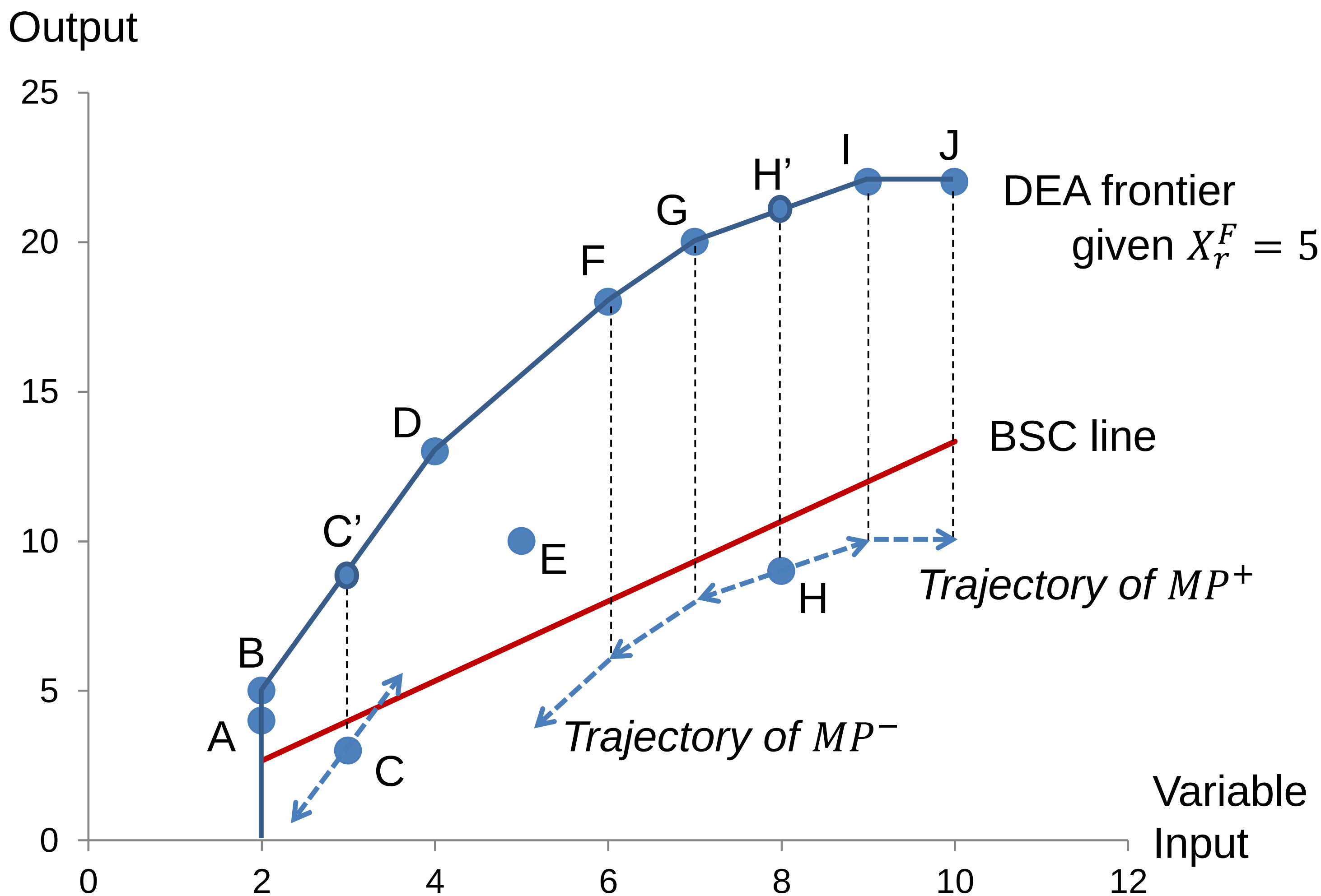
Business Shutdown Decision

- From a productivity and efficiency (PEA) perspective...



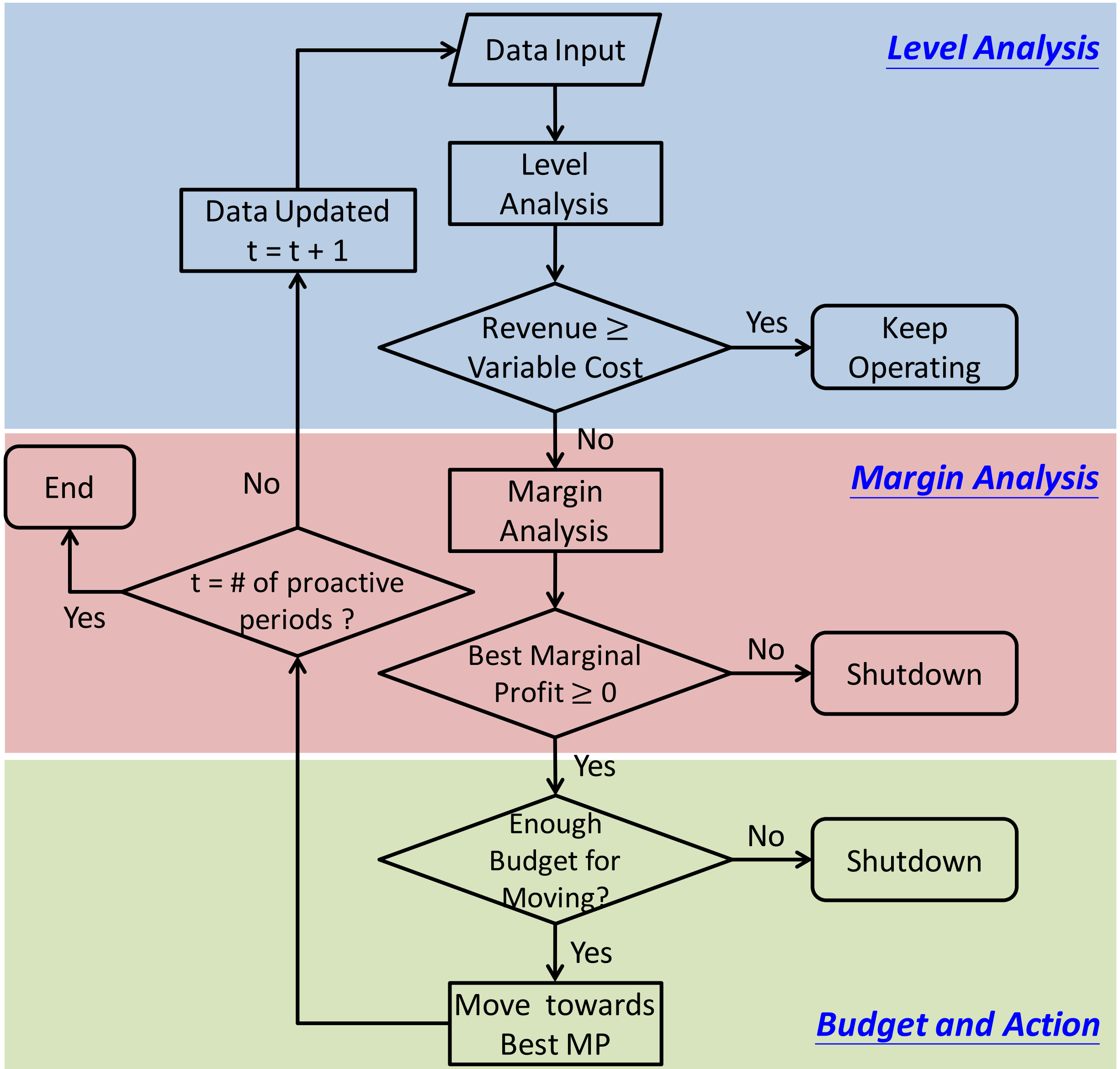
Business Shutdown Decision

- From a proactive PEA perspective...



Business Shutdown Decision

- Proposed BSC Scheme



Business Shutdown Decision

POLab

- An empirical study of 11 Taiwan LED manufacturers in 2012

Variable	Land & Building Assets	MOCVD		Die Output		Epi Wafer (6-inch)		Budget
Statistics	(Million US$)	Units	Operating Cost (US$M)	Die (10^6)	Price (US$)	Wafer (10^3)	Price (US$)	(US$M)
Mean	54.8	91.6	1.75	172.2	0.75	94.4	300	84.47
Std. ev.	58.4	74.4	0.84	214.7		131.1		139.11
Max.	218.2	288.8	3.20	765.6		367.7		472.81
Min.	12.3	17.9	0.75	26.7		0.0		-50.06

- For considering batch expansion in practice, we limit the positive and negative batch adjustment in MOCVD inputs to 15% in one year
- each MOCVD equipment costs US$2.5M.
- *budget and action*: Expected Working Capital

= Year-start + do-nothing + adjustment

= Current Working Capital + (Revenue – Variable Cost)

+ Marginal Profit × Input Adjustment

Business Shutdown Decision

- An empirical study of 11 Taiwan LED manufacturers in 2012

Firm	Level Analysis			Margin Analysis			Budget & Action		Shut-down
	Current Product-Mix (Y_1, Y_2)	Eff	Rev. -Vcost (US$M)	Best Direction (g_{Y_1}, g_{Y_2})	Exp or Con	Marginal Profit (US$M)	Current Working Capital (US$M)	Expected Working Capital (US$M)	
A	(0.98, 0.02)	1	90.50						
B	(0.965, 0,035)	1.17	1.77						
C	(1, 0)	1	-15.18	(0.2, 0.8)	Exp	4.32	2.13	-4.40	Y
	Keep operating in 2013 with net profit -164.45%								
D	(0.72, 0.28)	1	2.29						
E	(0.9, 0.1)	2.81	-12.76	(0.1, 0.9)	Exp	4.64	24.46	62.73	
	Keep operating in 2013 with net profit -44.96 %								
F	(0.995, 0.005)	3.75	-27.54	(0.1, 0.9)	Exp	5.34	57.18	109.72	
	Acquired by DMU A on Jan. 2013								
G	(0.5, 0.5)	1	47.78						
H	(0.74, 0.26)	1	46.03						
I	(0.98, 0.02)	1.40	-27.06	(0.1, 0.9)	Exp	3.87	-50.06		Y
	Shutdown July 2013								
J	(0.335, 0.665)	1	9.06						
K	(0.99, 0.01)	1.52	-14.00	(-1, 0)	Con	3.45	1.16	4.38	
	Keep operating in 2013 with net profit -51.0% and invested by American company								

- The DMP of Y_1 and Y_2 in DMU C

Case No. of Direction	Capacity Expansion: Direction (g_{Y_1}, g_{Y_2})	α	$\frac{\partial(Y_{Y_1}, Y_{Y_2})}{\partial X_{jr}^{V}}$	Marginal Profit (US$M)
Case 1	(1, 0)	0.036	(1.473, 0)	-3.47
Case 2	(0.9, 0.1)	0.040	(1.473, 0.811)	-3.23
Case 3	(0.8, 0.2)	0.045	(1.473, 1.826)	-2.93
Case 4	(0.7, 0.3)	0.052	(1.473, 3.130)	-2.53
Case 5	(0.6, 0.4)	0.060	(1.473, 4.869)	-2.01
Case 6	(0.5, 0.5)	0.073	(1.473, 7.303)	-1.28
Case 7	(0.4, 0.6)	0.091	(1.473, 10.954)	-0.19
Case 8	(0.3, 0.7)	0.121	(1.473, 17.040)	1.64
Case 9	**(0.2, 0.8)**	0.164	**(1.328, 26.343)**	**4.32**
Case 10	(0.1, 0.9)	0.145	(0.590, 26.343)	3.77
Case 11	(0, 1)	0.131	(0, 26.343)	3.32

Business Shutdown Decision

- Margin analysis of unchanged product-mix and returns to scale

Current Direction (unchanged product-mix)								
	Expansion			Contraction				
Firm	Current Direction (g_{Y_1}, g_{Y_2})	Marginal Profit (US$M)	Shut-down	Current Direction (g_{Y_1}, g_{Y_2})	Marginal Profit (US$M)	Expected Working Capital (US$M)	Shut-down	Returns to Scale (RTS)
C	(1, 0)	-3.47	Y	(-1, 0)	3.47	-6.10	Y	CRS
E	(0.9, 0.1)	-1.38	Y	(-0.9, -0.1)	1.38	26.89		IRS
F	(0.995, 0.005)	-1.59	Y	(-0.995, -0.005)	1.59	53.50		DRS
I	(0.98, 0.02)	-2.23	Y	(-0.98, -0.02)	2.23	-52.64	Y	IRS
K	(0.99, 0.01)	-3.45	Y	(-0.99, -0.01)	3.45	4.38		DRS

Business Shutdown Decision

Managerial Insights

- Clarify why a typical BSC level analysis could not identify a shutdown point effectively: (1) a firm's losses could be relatively small; (2) a firm could find a DMP direction to climb back above the BSC line; (3) a firm could try to wait out its competitors (i.e., finally returning to a break-even level). The results justified that "change is better than do nothing".
  - such as technology innovation, new product design, process improvement, and capacity adjustment.
- Given product-mix unchanged, the results suggest that firms should contract input resources and reduce the output level to avoid loss. Yet in 2013, 10 of the 11 LED manufacturers expanded capacity and produced more die outputs by adopting the business strategy that growing market share (i.e., being a "winner"). The phenomenon aligned with "the bigger the stronger, winner takes all" empirical evidence observed in the global high tech industry ecosystem (Hill and Jones, 2013).
- Furthermore, the expansion may also lead to another financial crisis in the short run since a long payback period of an expensive high-tech tool like MOCVD investment.

Business Shutdown Decision

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ORIGINAL ARTICLE

Proactive marginal productivity analysis for production shutdown decision by DEA

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ABSTRACT

The decision to shut a business usually arises when the product's marginal revenue falls below the average variable cost since the firm cannot offset the fixed cost. Today, however, this traditional business shutdown criterion (BSC) as defined by microeconomic theory may no longer apply to some types of industry; one example is the high-tech industry. This study proposes a BSC scheme that uses an iterative procedure embedded with three phases: level analysis, margin analysis, and budget and action, to solve the shutdown decision problem via proactive marginal productivity. We validate the proposed scheme with a case study of light-emitting diode manufacturers in Taiwan, the majority of which continued to operate and expand capacity despite experiencing a profit drop due to global competition and higher sunk cost of capital investment. Based on the results, we conclude that the proposed BSC scheme gives decision-makers an improved comprehensive prediction via margin analysis.

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KEYWORDS
Business shutdown; Data envelopment analysis (DEA); LED industry; marginal productivity; winner-takes-all;

1. Introduction

This study addresses the business shutdown criterion (BSC) based on productivity analysis and marginal productivity (MP). This is a new application line or exit the market. It is interesting to study the production behaviour of each firm in this system and investigate how they affect each other. The phenomenon motivated us to revisit the applicability of the BSC, in particular, the inadequacy of financial

Thanks for your attention!

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